Some Network-Theoretic Properties of Nonlinear DC Transistor Networks

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This paper extends, in several directions, some of the results of earlier work concerned with the existence and uniqueness of solutions of the dc equations of nonlinear transistor networks. In particular, here we develop techniques which enable us to deal directly with a more complicated transistor model.

I. INTRODUCTION

Several results are presented in Ref. 1 concerning the equation

$$F(x) + Ax = B \tag{1}$$

(with $F(\cdot)$ a "diagonal" nonlinear mapping of real Euclidean n-space E^n into itself, and A a real $n \times n$ matrix) which plays a central role in the dc analysis of transistor networks. In particular, a necessary and sufficient condition on A is given such that the equation possesses a unique solution x for each real n-vector B and each strictly monotone increasing $F(\cdot)$ that maps E^n onto itself. Several circuit-theoretic implications of the results are also described in Ref. 1; for example, it is shown that the short-circuit admittance matrix of the linear portion of the dc model of an interesting class of switching circuits must violate a certain dominance condition.

In Ref. 1 the word *transistor* was used to refer to the three-terminal device whose dc equivalent circuit is shown in Fig. 1(a). Although this equivalent circuit is frequently used in the design and computer analysis of transistor networks it is, from a physical standpoint, somewhat incomplete. A more exact dc model of a physical transistor is that of Fig. 1(b) in which the presence of series resistance in each of the transistor's leads has been accounted for.

In this paper we report on several extensions of the previous results. The motivation for much of this work was to enable the model of Fig. 1(b) to be taken into account. In addition, we present here

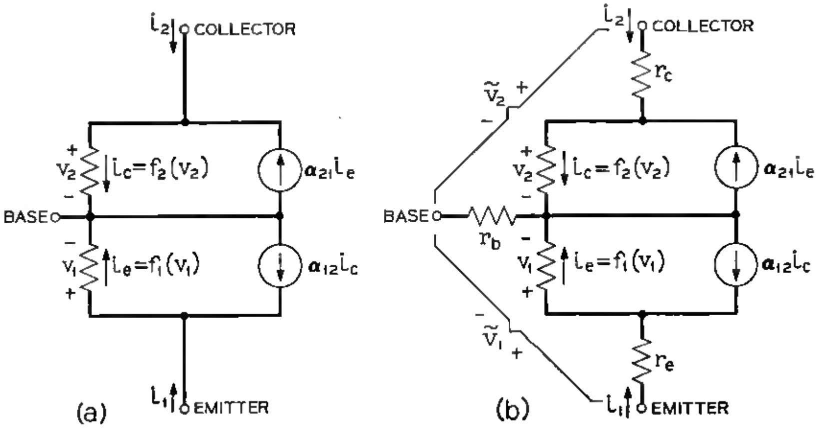

Fig. 1 — DC transistor models.

further material concerning cases in which (in accordance with standard assumptions) the nonlinear functions of Fig. 1(b) do not map E^1 *onto* itself. Finally, we prove a considerably stronger result than that of Ref. 1, to the effect that a certain class of networks cannot be bistable.

We now summarize some of the material of Ref. 1 that will be needed in the sequel:

For each positive integer n, we let $\mathfrak{F}^n$ denote that collection of mappings of the real n-dimensional Euclidean space E^n onto itself defined by: $F \in \mathfrak{F}^n$ if and only if there exist, for $i = 1, \cdots, n$, strictly monotone increasing functions f_i mapping E^1 onto E^1 such that,† for each $x \equiv (x_1, \cdots, x_n)' \in E^n$, $F(x) \equiv (f_1(x_1), \cdots, f_n(x_n))'$.

The origin in E^n will be denoted by θ. Throughout this article we consider only matrices whose elements are real. If D is a diagonal matrix then $D > 0$ ($D \geqq 0$) means that each element on the main diagonal of D is positive (nonnegative).

The classes of matrices P and P_0 have been defined by M. Fiedler and V. Pták in Refs. 2 and 3. They prove that these classes can be defined by any one of several equivalent properties. We shall need only the following characterization of the classes P and P_0: A square matrix A is a member of the class P (P_0) if and only if all principal minors of A are positive (nonnegative). In the appendix it is proved that $A \in P_0$ if and only if $\det [A + D] \neq 0$ for every diagonal matrix $D > 0$.

† If M is an arbitrary matrix, then the transpose of M is denoted in this article by M'.

The following theorem is proved in Ref. 1:

Theorem 1: If A is an $n \times n$ matrix then there exists a unique solution of (1) *for each $F \in \mathfrak{F}^n$ and each $B \in E^n$ if and only if $A \in P_0$.*

We say that an $n \times n$ matrix A is *strongly (weakly) row-sum dominant* if and only if the elements a_{ij} of A satisfy

$$a_{ii} > (\geqq) \sum_{j \neq i} |a_{ij}|, \quad \text{for} \quad i = 1, \cdots, n.$$

Similarly, a *strongly (weakly) column-sum dominant* matrix is one that satisfies

$$a_{ii} > (\geqq) \sum_{j \neq i} |a_{ji}|, \quad \text{for} \quad i = 1, \cdots, n.$$

The square matrix A is said to be *dominant* (*strongly dominant*) if and only if A is weakly (strongly) row-sum dominant and symmetric.

If a square matrix A is strongly column-sum or row-sum dominant then A is nonsingular, in fact $A \in P$.

The following theorem is also proved in Ref. 1:

Theorem 2: If the square matrix A satisfies a strong column-sum dominance condition and if the square matrix B satisfies a weak (*strong*) *column-sum dominance condition, then $A^{-1}B \in P_0$* (*P*).

An analogous theorem involving row-sum dominant matrices is also true, and can be proved with trivial modifications of the proof of Theorem 2 given in Ref. 1.

II. FURTHER RESULTS CONCERNING THE EXISTENCE AND UNIQUENESS OF SOLUTIONS

The proof of Theorem 1 given in Ref. 1 exploits the fact that the straight line described by the equation $y = -ax + b$ has exactly one intersection with the graph of each strictly monotone increasing function $f(x)$ which maps E^1 onto E^1 if and only if $a \geqq 0$.

It happens that a useful result that is slightly more general than that of Theorem 1 can be proved easily if use is made of a proposition that is similar to, but stronger than, the elementary fact mentioned in the preceding paragraph. That proposition is stated below.

Definition: For all α, β with $-\infty \leqq \alpha < \beta \leqq \infty$, let $I(\alpha, \beta)$ denote the interval $I(\alpha, \beta) = \{x : \alpha < x < \beta\}$.

The following proposition is quite easily verified:

Proposition: For $-\infty \leqq \alpha < \beta \leqq \infty$, the straight line described by the

equation $y = -ax + b$ has exactly one intersection with the graph of each strictly monotone increasing function $f(x)$ which maps $I(\alpha, \beta)$ onto E^1 if and only if $a \geqq 0$.

Definition: For each positive integer n and each pair of n-vectors α, β whose components α_i, β_i lie in the extended real number system, with $\alpha < \beta$ (that is, with $-\infty \leqq \alpha_i < \beta_i \leqq \infty$ for $i = 1, \cdots, n$) let $\mathfrak{F}^n(\alpha, \beta; E^n)$ denote that collection of mappings of $I(\alpha_1, \beta_1) \times \cdots \times I(\alpha_n, \beta_n)$ onto E^n defined by: $F \in \mathfrak{F}^n(\alpha, \beta; E^n)$ if and only if there exist, for $i = 1, \cdots, n$, strictly monotone increasing functions f_i mapping (α_i, β_i) onto E^1 such that for each $x \equiv (x_1, \cdots, x_n)^t \in I(\alpha_1, \beta_1) \times \cdots \times I(\alpha_n, \beta_n)$,

$$F(x) \equiv (f_1(x_1), \cdots, f_n(x_n))^t.$$

Let the collection of strictly monotone increasing mappings of E^n onto $I(\alpha_1,\beta_1) \times \cdots \times I(\alpha_n,\beta_n)$ be similarly defined, and denoted by $\mathfrak{F}^n(E^n; \alpha, \beta)$. Note that $F \in \mathfrak{F}^n(\alpha, \beta; E^n)$ if and only if F^{-1} exists and $F^{-1} \in \mathfrak{F}^n(E^n; \alpha, \beta)$. Also, in case $I(\alpha_1,\beta_1) \times \cdots \times I(\alpha_n,\beta_n) = E^n$, then $\mathfrak{F}^n(\alpha, \beta; E^n) = \mathfrak{F}^n(E^n; \alpha, \beta) = \mathfrak{F}^n$.

Using the above proposition it is now easy to prove:

Theorem 3: For the n-vectors $\alpha < \beta$ whose components lie in the extended real number system, if A is an $n \times n$ matrix then there exists a unique solution of (1) *for each $F \in \mathfrak{F}^n(\alpha, \beta; E^n)$ and each $B \in E^n$ if and only if $A \in P_0$.*

Proof: (*if*) The proof of this part of the theorem is identical to the proof (given in Ref. 1) of the corresponding part of Theorem 1 with the exception that appropriate use is made of the above proposition. Since the necessary modifications are quite obvious we omit the details.

(*only if*) Suppose $A \notin P_0$. Then there exists a diagonal matrix $D \equiv \operatorname{diag}[d_1, \cdots, d_n] > 0$ such that $\det[A + D] = 0$. Let x^0 be an arbitrary point in $I(\alpha_1, \beta_1) \times \cdots \times I(\alpha_n, \beta_n)$ and let y^0 be an arbitrary point in E^n. Let

$$B = y^0 + Ax^0.$$

Let $\delta > 0$ be chosen such that

$$\alpha_i < x_i^0 - \delta < x_i^0 + \delta < \beta_i, \qquad \text{for } i = 1, \cdots, n,$$

and choose $F \equiv (f_1(\cdot), \cdots, f_n(\cdot))^t$ in $\mathfrak{F}^n(\alpha, \beta; E^n)$ such that for $i = 1, \cdots, n$, and for $x_i^0 - \delta < x_i < x_i^0 + \delta$,

$$f_i(x_i) = y_i^0 + d_i(x_i - x_i^0).$$

Thus, $F(x^0) = y^0$ and hence, x^0 is a solution of (1) for this choice of F.

Since det $[A + D] = 0$, there exists some n-vector $x^* \neq \theta$ having the property that

$$Ax^* + Dx^* = \theta.$$

Thus, for each real numher ϵ,

$$y^0 + D\epsilon x^* + A(x^0 + \epsilon x^*) = B.$$

In particular, if $\epsilon \neq 0$ is chosen such that $|\epsilon|$ is sufficiently small, then $| \epsilon x_i^* | < \delta$ for $i = 1, \cdots, n$. Hence, for such ϵ, if $x = x^0 + \epsilon x^*$, $F(x) = y^0 + D\epsilon x^*$ and therefore $x \neq x^0$ is also a solution of (1). □

An important special case of Corollary 3 of Ref. 1 is:

Corollary 1: *For the n-vectors $\alpha < \beta$ whose components lie in the extended real number system, if A is an $n \times n$ motrix then there exists a unique solution of* (1) *for each $F \in \mathfrak{F}^n(E^n; \alpha, \beta)$ and each $B \in E^n$ if $A \in P$.*

Theorem 3 may be used to prove a sharper (and, from the viewpoint of transistor networks, a more useful) result than Corollary 1. We have:

Theorem 4: For the n-veetors $\alpha < \beta$ whose components lie in the extended real number system (in the real number system), if A is an $n \times n$ matrix then there exists a unique solution of (1) *for eaeh $F \in \mathfrak{F}^n(E^n; \alpha, \beta)$ and each $B \in E^n$ if (and only if) $A \in P_0$ and det $A \neq 0$.*

Proof: (*if*) As pointed out in Ref. 1, $A \in P_0$ and det $A \neq 0$ imply that $A^{-1} \in P_0$. Also, F^{-1} exists and $F^{-1} \in \mathfrak{F}^n(\alpha, \beta; E^n)$. Now x satisfies (1) if and only if y satisfies

$$F^{-1}(y) + A^{-1}y = A^{-1}B, \tag{2}$$

where $y = F(x)$. But, according to Theorem 3, there exists a unique y which satisfies (2).

(*only if*) We assume here that the components of α and β are real. Suppose $A \notin P_0$. Then, in a manner similar to that used in the proof of the "only if" part of Theorem 3, we can choose a mapping $F \in \mathfrak{F}^n(E^n; \alpha, \beta)$ and a point $B \in E^n$, such that the solution of (1) is not unique.

If, on the other hand, det $A = 0$, then there exists $x^* \neq \theta$ such that $A^t x^* = \theta$. Assume that (1) has a solution x for each $B \in E^n$. Then, since $\langle x^*, Ax\rangle = 0$ for all x, we have

$$\langle x^*, F(x)\rangle = \langle x^*, B\rangle,$$

for each $B \in E^n$ (and the corresponding x). It is clear, since the com-

ponents of α and β are finite, that there exists some constant M such that

$$|\langle x^*, F(x)\rangle| \leqq M$$

for all $x \in E^n$. But B can certainly be chosen such that $\langle x^*, B\rangle > M$. This contradiction completes the proof of the theorem. □

The following theorem provides an alternative method of characterizing the class of matrices that are in P_0 and are nonsingular (compare with the theorem of the appendix).

Theorem 5: If A is a real square matrix then $A \in P_0$ and det $A \neq 0$ if and only if det $[A + D] \neq 0$ for every diagonal matrix $D \geqq 0$.

Proof: (*if*) It is clear, by the theorem of the appendix, that $A \in P_0$, since det $[A + D] \neq 0$ for all diagonal $D > 0$. Moreover, det $A \neq 0$, by hypothesis.

(*only if*) It is shown in Ref. 1 that, for each $A \in P_0$ and each diagonal $D \geqq 0$, $A + D \in P_0$. It suffices, therefore, to show that if $D_i = \text{diag}\,[0, \cdots, 0, d_i, 0, \cdots, 0]$ with $d_i \geqq 0$, and $A \in P_0$ with det $A > 0$, then det $[A + D_i] > 0$. Letting A_i denote the principal submatrix obtained from A by deleting the ith row and the ith column, we have

$$\det\,[A + D_i] = \det A + d_i \det A_i\,.$$

But det $A > 0$ and d_i det $A_i \geqq 0$. □

III. APPLICATION TO EQUATIONS FOR TRANSISTOR NETWORKS

In the analysis of a transistor network one could account for the presence of series lead resistance, while using the model of Fig. 1(a) to represent the transistor, by including appropriate additional resistors in the rest of the network. Indeed, there is at least one good reason for doing this. When treated in this manner, the presence of nonzero series resistance in the base, collector, and emitter leads of each transistor ensures that the y-parameter matrix exists for the circuit to which the transistors are connected—and hence ensures that the transistor network can be described by an equation having the form of (1). On the other hand, there are also good reasons for representing the transistor, for analysis purposes, by the model of Fig. 1(b). Using this model it will be shown, for example, that it is often possible to determine that there is a unique solution of the equation describing a given transistor network *regardless* of the (nonnegative)

values of the transistors' series lead resistances. Since these resistances are usually parasitic and unavoidable in nature it is significant that one might be able to show that their introduction in, say, a certain monostable circuit will not cause the circuit to become bistable.

Using the model of Fig. 1(b) it is quite easy to see that the port variables for the transistor, when considered as a nonlinear two-port network, obey the following relationship

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 & -\alpha_{12} \\ -\alpha_{21} & 1 \end{bmatrix} \begin{bmatrix} f_1(v_1) \\ f_2(v_2) \end{bmatrix}$$

where

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} - \begin{bmatrix} r_e + r_b & r_b \\ r_b & r_c + r_b \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}.$$

As in Ref. 1 we assume that $0 < \alpha_{12} < 1$, $0 < \alpha_{21} < 1$, and that both of the functions f_1 and f_2 are strictly monotone increasing mappings of E^1 into E^1.

Suppose an electrical network is synthesized containing transistors, resistors (that is, linear resistors having nonnegative resistance), independent voltage and current sources, and nonlinear resistors which are described by strictly monotone increasing conductance functions (and which shall henceforth be called "diodes"). Suppose the network contains n transistors and d diodes $(n + d > 0)$. For $k = 1, \cdots, n$ let x_{2k-1}, x_{2k}, $\bar{x}_{2k-1}$, $\bar{x}_{2k}$, y_{2k-1}, and y_{2k} denote the voltage and current variables v_1, v_2, $\bar{v}_1$, $\bar{v}_2$, i_1, and i_2, respectively, for the kth transistor. For $k = 1, \cdots, d$, let x_{2n+k} and y_{2n+k} denote the voltage across, and the current through, the kth diode; also (for $k=1, \cdots, d$) let $\bar{x}_{2n+k} = x_{2n+k}$. Let these variables be related by $y_{2n+k} = f_{2n+k}(x_{2n+k})$. Then, if $x = (x_1, \cdots, x_{2n+d})^t$, $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_{2n+d})^t$, and $y = (y_1, \cdots, y_{2n+d})^t$, we have

$$y = TF(x), \qquad x = \bar{x} - Ry, \tag{3}$$

where $T = \operatorname{diag}[T_1, T_2]$, with T_1 a block diagonal matrix with n 2×2 diagonal blocks of the form

$$\begin{bmatrix} 1 & -\alpha_{12}^{(k)} \\ -\alpha_{21}^{(k)} & 1 \end{bmatrix}, \tag{4}$$

and T_2 the $d \times d$ identity matrix. Also, $R = \operatorname{diag}[R_1, R_2]$, with R_1 a block diagonal matrix with n 2×2 diagonal blocks of the form

$$\begin{bmatrix} r_e^{(k)} + r_b^{(k)} & r_b^{(k)} \\ r_b^{(k)} & r_c^{(k)} + r_b^{(k)} \end{bmatrix}, \tag{5}$$

and R_2 the $d \times d$ null matrix.

Consider now the $(2n + d)$-port network of resistors and independent sources which is formed from the original network by removing the transistors and diodes. If the y-parameter matrix G of this $(2n + d)$-port exists then we have the additional equation relating the vectors $\tilde{x}$ and y:

$$y = -G\tilde{x} + \tilde{u} \tag{6}$$

where $\tilde{u}$ is some vector of constants that is, in general, nonzero since sources are present in the $(2n + d)$-port.

The vectors $\tilde{x}$ and y can easily be eliminated from (3) and (6), resulting in the equation

$$TF(x) + [I + GR]^{-1}Gx = u, \tag{7}$$

where we have defined the vector u by

$$u = [I + GR]^{-1}\tilde{u}.$$

According to Theorem 6, below, the matrix $[I + GR]$ must be nonsingular.

In case the matrix R contains all zeros (that is, in case all series lead resistors are omitted from the transistors) (7) reduces immediately to the equation which was studied in Ref. 1. Even when R does not contain all zeros, however, the results of Ref. 1 can be applied directly to (7). By applying Theorem 2 we have: *If the matrix* $[I + GR]^{-1} G$ *is dominant*† *then there is at most one solution of* (7). *If, furthermore,* F *maps* E^n *onto* E^n, *or if* $[I + GR]^{-1}G$ *is strongly dominant, then there exists a unique solution of* (7).

Making use of Theorem 4, we also have the stronger result: *There exists a unique solution of* (7) *if* $[I + GR]^{-1}G$ *is dominant and* G *is nonsingular.*

Although it is not, in general, true that the inverse of a strongly column-sum (row-sum) dominant matrix is strongly row-sum (column-sum) dominant, the statement is true when the order of the matrix is less than three. This elementary observation turns out to be quite useful in the proof of Theorem 6, which yields results that focus attention on the properties of G, concerning the existence and uniqueness of a solution of (7).

† For symmetric matrices the properties (i) weak column-sum dominance, and (ii) dominance, are identical. Since it is easily verified that for symmetric G and R, $[I + GR]^{-1}G$ is also symmetric, we simply specify that $[I + GR]^{-1}G$ be dominant.

Theorem 6: Let A (B) be the direct sum of n 2×2 and d 1×1 strongly column-sum (weakly row-sum) dominant matrices. Let B be symmetric and let C be a square matrix of order $2n + d$. Then:

(*i*) $\det [I + CB] \neq 0$, *provided that C is positive semidefinite,*
(*ii*) $A^{-1}[I + CB]^{-1}C \in P_0$, *provided that C is dominant,*
(*iii*) $A^{-1}[I + CB]^{-1}C \in P$, *provided that C is strongly dominant.*

Proof: (*i*) Here C is positive semidefinite. Let $B^{\frac{1}{2}}$ be the symmetric nonnegative square root of B, so that $I + CB = I + CB^{\frac{1}{2}}B^{\frac{1}{2}}$. Since (see Appendix A of Ref. 4) $\det [I + CB^{\frac{1}{2}}B^{\frac{1}{2}}] = \det [I + B^{\frac{1}{2}}CB^{\frac{1}{2}}]$, and since $I + B^{\frac{1}{2}}CB^{\frac{1}{2}}$ is positive definite, we have $\det [I + CB] > 0$.

(*ii*) Here C is dominant (which, as is well known, implies that C is positive semidefinite and hence, by (*i*), implies that $[I + CB]^{-1}$ exists). Suppose $A^{-1}[I + CB]^{-1}C \notin P_0$. Then, by the theorem of the appendix, there exists a diagonal matrix $D > 0$ such that $A^{-1}[I + CB]^{-1}C + D$ is singular. But

$$A^{-1}[I + CB]^{-1}C + D = A^{-1}[I + CB]^{-1}[C(D^{-1}A^{-1} + B) + I]AD,$$

which means that $C(D^{-1}A^{-1} + B) + I$ must be singular. Since A is a direct sum of 1×1 and 2×2 strongly column-sum dominant matrices, it follows that A^{-1} is a direct sum of 1×1 and 2×2 strongly row-sum dominant matrices. Thus, $D^{-1}A^{-1}$ and hence $D^{-1}A^{-1} + B$ is strongly row-sum dominant. Therefore, $(D^{-1}A^{-1} + B)$ is nonsingular, and $(D^{-1}A^{-1} + B)^{-1}$ is strongly column-sum dominant. But,

$$C(D^{-1}A^{-1} + B) + I = [C + (D^{-1}A^{-1} + B)^{-1}](D^{-1}A^{-1} + B)$$

in which the right-hand side is nonsingular since $C + (D^{-1}A^{-1} + B)^{-1}$ is strongly column-sum dominant, which is a contradiction.

(*iii*) Here C is strongly dominant. Since $C(I + BC) = (I + CB)C$, we have $\det(I + BC) > 0$ and

$$(I + CB)^{-1}C = C(I + BC)^{-1}.$$

Suppose that there is no constant $\delta > 0$ such that $A^{-1}C(I + BC)^{-1} - \delta I \in P_0$. Then, for each $\delta > 0$ there is a diagonal matrix $D > 0$ such that $A^{-1}C(I + BC)^{-1} - \delta I + D$ is singular. But,

$$\begin{aligned} A^{-1}C(I + BC)^{-1} - \delta I + D \\ &= A^{-1}[C - \delta A(I + BC) + AD(I + BC)](I + BC)^{-1} \\ &= D\{I + BC + D^{-1}A^{-1}[C - \delta A(I + BC)]\}(I + BC)^{-1} \\ &= \{D + [DB + A^{-1} - \delta(C^{-1} + B)]C\}(I + BC)^{-1}, \end{aligned}$$

which leads to the conclusion that for each $\delta > 0$ there is a $D > 0$ such that $D + [DB + A^{-1} - \delta(C^{-1} + B)]C$ is singular. We now establish a contradiction:

For all $x \in E^n$, let $\| x \| = \max_i | x_i |$. If $x, y \in E^n$ such that $\| x \| = 1$ and

$$[DB + A^{-1}]y = x$$

then it is easy to show that

$$\| y \| \leqq \max_k \frac{1}{\alpha_{kk} - \sum_{j \neq k} | \alpha_{kj} |}$$

in which the α_{kj} are the elements of A^{-1}. Thus, the norm of $[DB + A^{-1}]^{-1}$ can be hounded from above uniformly in $D > 0$. Therefore,

$$D + [DB + A^{-1} - \delta(C^{-1} + B)]C = (DB + A^{-1})\{(DB + A^{-1})^{-1}D + [I - \delta(DB + A^{-1})^{-1}(C^{-1} + B)]C\}$$

in which $\delta > 0$ can be chosen so small that $[I - \delta(DB + A^{-1})^{-1}(C^{-1} + B)]C$ is strongly column-sum dominant for all $D > 0$. Since $(DB + A^{-1})^{-1}D$ is also column-sum dominant, we have a contradiction. It follows that for some $\delta > 0$, $A^{-1}(I + CB)^{-1}C - \delta I \in P_0$ and hence, by Theorem 1 of Ref. 1, $A^{-1}(I + CB)^{-1}C \in P$. □

The matrices T, R, and G of (7) satisfy the hypotheses on A, B, and C, respectively, of Theorem 6 if it happens that G is dominant (strongly dominant for (*iii*)). Thus, we have the result: *If the y-parameter matrix G is dominant then there is at most one solution of* (7). *If, furthermore, F maps E^n onto E^n, or if G is strongly dominant, then there exists a unique solution of* (7).

Making use of Theorem 4 and since det $C \neq 0$ implies det $[A^{-1}(I + CB)^{-1}C] \neq 0$, we also have: *There exists a unique solution of* (7) *if G is dominant and nonsingular.*

These results show that if the solution of the equation

$$TF(x) + Gx = \tilde{u}, \tag{8}$$

describing a given transistor network (with the transistors represented by the model of Fig. 1(a) is shown to (exist and) he unique by showing that the y-parameter matrix G is dominant (and det $G \neq 0$, or that F maps E^n onto E^n), then any other network obtained from the original by adding arbitrary (nonnegative) resistances in series with any of the transistor leads will be described hy (7) and, furthermore, the solution of (7) will also (exist and) be unique. Thus, the addition of series lead

resistance does not affect the existence and uniqueness of the solution, provided G is dominant.

We now prove another result concerning the relationship between the existence and uniqueness of solutions of the two equations (7) and (8). We prove that, roughly speaking, whenever (8) has a unique solution for all transistors and diodes then so does (7). More precisely, let us define, for a given transistor network, the class of matrices $\mathfrak{J}$:

Definition: Let (8) describe the given network for some choice of transistor parameters α_{12}, α_{21}, for each transistor. Let $\mathfrak{J}$ then denote that class of matrices T obtained by considering all possible combinations of values of α_{12}, α_{21} ($0 < \alpha_{12} < 1, 0 < \alpha_{21} < 1$) for each transistor.

We then have:

Theorem 7: If (8) *has a unique solution for each* $T \in \mathfrak{J}$, *and each* $F \in \mathfrak{F}^n(E^n; \alpha, \beta)$ *for all* $\alpha < \beta$ *whose components lie in the extended real number system then, for each* R, *so does* (7).

Proof: The hypotheses imply (using Theorem 4) that $T^{-1}G \in P_0$ and $\det [T^{-1}G] \neq 0$ for each $T \in \mathfrak{J}$. Thus, G^{-1} exists. Letting

$$H \equiv [I + GR]^{-1}G,$$

H^{-1} exists and,

$$H^{-1} = G^{-1} + R.$$

As pointed out in Ref. 1, since $\det [T^{-1}G] \neq 0$, $T^{-1}G \in P_0$ for every $T \in \mathfrak{J}$ implies that $G^{-1}T \in P_0$ for every $T \in \mathfrak{J}$. Hence

$$\det [G^{-1}T + D] > 0, \qquad \text{for all } T \in \mathfrak{J} \text{ and all } D > 0.$$

But then,

$$\det [G^{-1} + DT^{-1}] > 0, \qquad \text{for all } T \in \mathfrak{J} \text{ and all } D > 0.$$

Now, due to the special structure of the matrix R (that is, block diagonal with dominant blocks that are "compatible" with T^{-1}) it is clear that, for any such R, any diagonal $D > 0$, and any $T \in \mathfrak{J}$, there exists a diagonal $\Delta > 0$ and some $M \in \mathfrak{J}$, such that $R + DT^{-1} = \Delta M^{-1}$. Hence, it is clear that

$$\det [G^{-1} + R + DT^{-1}] > 0, \qquad \text{for all } T \in \mathfrak{J} \text{ and all } D > 0.$$

It easily follows that $H^{-1}T \in P_0$ and hence $T^{-1}H \in P_0$ for all $T \in \mathfrak{J}$. Applying Theorem 4, we thus have that there exists a unique solution

of (7) for each $T \varepsilon \mathfrak{I}$, and each $F \varepsilon \mathfrak{F}^n(E^n; \alpha, \beta)$ for all $\alpha < \beta$ whose components lie in the extended real numher system. □

It is not difficult to show that there exist transistor networks for which $[I + GR]^{-1}G$ is dominant while G is not, and also networks for which G is dominant while $[I + GR]^{-1}G$ is not. For the first case, pick any network for which G is not dominant and det $G \neq 0$. If the values of the series lead resistors in each transistor lead are then allowed to hecome large, since

$$[I + GR]^{-1}G = [I + R^{-1}G^{-1}]^{-1}R^{-1},$$

and since each element of R^{-1} approaches zero as the lead resistor values approach infinity, we see that $[I + GR]^{-1}G \to R^{-1}$. But R^{-1} is strongly dominant and hence there certainly exist sufficiently large values for the lead resistors such that $[I + GR]^{-1}G$ is dominant. The network of Fig. 2 is an example of the other case. For this network,

$$G = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 9 & 9 & 0 & 0 \\ 9 & 9 & 0 & 0 \\ 0 & 0 & 9 & 9 \\ 0 & 0 & 9 & 9 \end{bmatrix},$$

while

$$[I + GR]^{-1}G = \frac{1}{37}\begin{bmatrix} 19 & -18 & -19 & 18 \\ -18 & 19 & 18 & -19 \\ -19 & 18 & 19 & -18 \\ 18 & -19 & -18 & 19 \end{bmatrix}.$$

IV. A SPECIAL CLASS OF TRANSISTOR NETWORKS

Transistor networks in which the hase terminal of each transistor is connected to a common node are considered in Ref. 1 using the model of Fig. 1(a) to represent the transistor. It is shown there that there is at most one pair of hase-collector and hase-emitter voltages for each transistor in such a network—even in the cases in which the network is not descrihed hy an equation having the form of (1).

In this section we show that the class of common-hase transistor networks is hut a suhset of a considerahly more extensive special class of transistor networks for which the same statement is true. We show that there is at most one pair of hase-collector and hase-emitter volt-

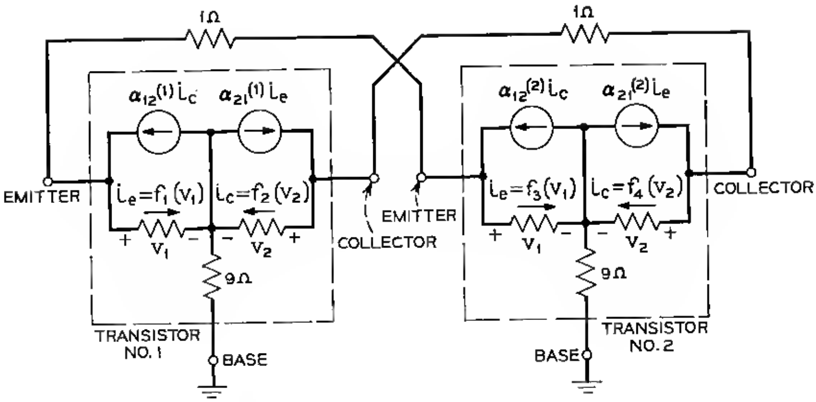

Fig. 2 — A two-transistor network.

ages for each transistor in any dc network which has the structure shown in Fig. 3. The box at the top of Fig. 3 represents, assuming that there are n transistors, any $(2n + 1)$-terminal network consisting of independent voltage and current sources, resistors (that is, linear resistors having nonnegative resistance), and diodes (that is, nonlinear resistors which are described by strictly monotone increasing conductance functions). Each of the n boxes at the bottom of Fig. 3 represents an

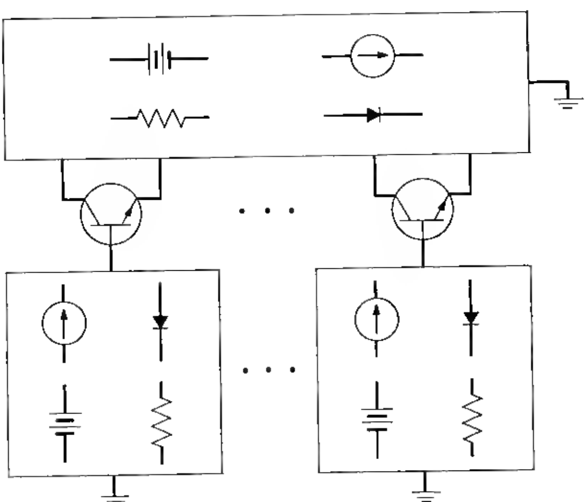

Fig. 3 — A special class of transistor networks.

arbitrary 2-terminal network consisting of independent sources, resistors, and diodes. Each of the transistors in Fig. 3 is represented by the model of Fig. 1(b), in which the value of each of the resistors r_b, r_c, r_e may be any nonnegative number. In this regard, we note here that it suffices in what follows to show, for each transistor, the uniqueness of the voltages v_1 and v_2 (in Fig. 1(b)) since, clearly, the voltages $\tilde{v}_1$ and $\tilde{v}_2$ are then uniquely determined.

As in Ref. 1 we assume, temporarily, that no diodes are present in the network. This assumption allows each of the n boxes at the bottom of Fig. 3 to be replaced by either a current source or else a Thévenin's equivalent circuit in which the value of the Thévenin's resistor is not infinite. Let us temporarily ignore the possibility that any of these boxes is equivalent to a current source. Following the technique presented in Section IX of Ref. 1, we may then consider the network of Fig. 4 instead of that of Fig. 3. In Fig. 4 we have explicitly shown the base, emitter, and collector resistors of each transistor, and we consider the Thévenin's resistor of each base circuit to be lumped in with the corresponding base resistor. The m-vectors v^* and i^* $(m \leqq 2n)$ and the $2n$-vectors v' and i' are related by the four equations:

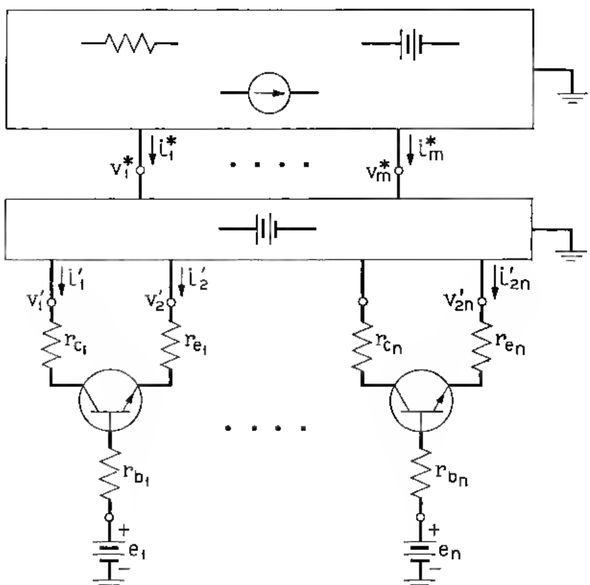

Fig. 4 — Network derived from that of Fig. 3.

$$i^* = -Gv^* + b, \tag{9}$$

$$i^* = Qi', \tag{10}$$

$$v' = Q^t v^* + c, \tag{11}$$

$$i' = TF(v' - e - Ri'), \tag{12}$$

in which b, c, and e are vectors whose elements are constants, G is a dominant matrix, Q is an $m \times 2n$ matrix having the property that whenever the $2n \times 2n$ matrix M is strongly column-sum dominant then so is the $m \times m$ matrix QMQ^t, T and R are $2n \times 2n$ block diagonal matrices having 2×2 diagonal blocks of the form (4) and (5), respectively.

We now show that the vectors v^*, i^*, v', and i' which satisfy (9) through (12) are unique (if they exist). Let $\{v^*_{(1)}, i^*_{(1)}, v'_{(1)}, i'_{(1)}\}$ and $\{v^*_{(2)}, i^*_{(2)}, v'_{(2)}, i'_{(2)}\}$ denote two sets of vectors, each of which satisfies (9) through (12). Subtracting corresponding equations, and observing the strictly monotone character of F, we see that there exists a diagonal matrix $D > 0$ such that:

$$i^*_{(1)} - i^*_{(2)} = -G(v^*_{(1)} - v^*_{(2)}), \tag{13}$$

$$i^*_{(1)} - i^*_{(2)} = Q(i'_{(1)} - i'_{(2)}), \tag{14}$$

$$v'_{(1)} - v'_{(2)} = Q^t(v^*_{(1)} - v^*_{(2)}), \tag{15}$$

$$i'_{(1)} - i'_{(2)} = TD(v'_{(1)} - v'_{(2)} - R(i'_{(1)} - i'_{(2)})). \tag{16}$$

But (15) and (16) imply

$$[I + TDR](i'_{(1)} - i'_{(2)}) = TDQ^t(v^*_{(1)} - v^*_{(2)}).$$

However, since

$$[I + TDR] = T[T^{-1} + DR],$$

in which T is strongly column-sum dominant (T^{-1} is strongly row-sum dominant), and DR is weakly row-sum dominant, we have $\det [I + TDR] \neq 0$, and hence,

$$i'_{(1)} - i'_{(2)} = [I + TDR]^{-1}TDQ^t(v^*_{(1)} - v^*_{(2)}). \tag{17}$$

Substituting this into (14) and then (13), however, yields:

$$\{Q[I + TDR]^{-1}TDQ^t + G\}(v^*_{(1)} - v^*_{(2)}) = \theta.$$

Now if $Q[I + TDR]^{-1}TDQ^t + G$ can be shown to be nonsingular then $v^*_{(1)} - v^*_{(2)} = \theta$ and hence, by (13), (15), and (17): $i^*_{(1)} - i^*_{(2)} = \theta$, $v'_{(1)} - v'_{(2)} = \theta$, and $i'_{(1)} - i'_{(2)} = \theta$, which, together, show that the

vectors which satisfy (9) through (12) are unique. Since G is dominant it suffices to show that $[I+TDR]^{-1}TD$ (and hence $Q[I+TDR]^{-1}TDQ'$) is strongly column-sum dominant. But

$$[I + TDR]^{-1}TD = [D^{-1}T^{-1} + R]^{-1},$$

which is the inverse of the direct sum of 2×2 strongly row-sum dominant matrices and is, therefore, strongly column-sum dominant.

Let us now consider the case in which diodes are present in the box at the top of Fig. 3. In this case, arguing as in Section IX of Ref. 1, if the set of base-emitter and base-collector voltages for Fig. 3 was not unique, we could replace all of the diodes by an appropriate series combination of a voltage source and a (nonnegative) resistor and thus synthesize a network of the type just considered, for which the set of base-emitter and base-collector voltages is not unique. This is a contradiction, and hence establishes that the set of base-emitter and base-collector voltages for the network of Fig. 3 is unique even when diodes are present in the top box.

A somewhat similar argument may now be used to show the uniqueness of the voltage across each of the diodes in the box at the top of Fig. 3. Assume that there exist two sets of branch voltages and currents, S_1 and S_2, which satisfy Kirchoff's and Ohm's laws for the network of Fig. 3. Since we have just proved the uniqueness of the base-emitter and base-collector voltages of each transistor, the elements of S_1 and S_2 which correspond to any such voltage must be identical. Thus, if each transistor is replaced by, say, an appropriate pair of voltage sources, the sets S_1 and S_2 still satisfy Kirchoff's and Ohm's laws for the modified network. Let us now choose (arbitrarily) any diode in the network and, as in the previous argument, replace all other diodes by a series combination of a voltage source and a (nonnegative) resistor, thus obtaining a new network, containing only one diode, for which the sets S_1 and S_2 still satisfy Kirchoff's and Ohm's laws. Suppose this remaining diode is characterized by the equation $i = f(v)$. The (now linear) network to which this diode is connected contains only independent sources and nonnegative resistors, and hence is characterized by one of the equations: $-i = gv + I_0$, $v = V_0$, where $g \geqq 0$, I_0, and V_0 are constants. Due to the strictly monotone increasing character of f, however, the graph of either of the above equations can intersect the graph of f in at most one point. Thus, the elements of S_1 and S_2 that specify the voltage across this diode must be equal. We can therefore conclude that the corresponding elements of S_1 and S_2 which specify the voltage across any

diode are equal. That is, the diode voltages are unique for all diodes in the hox at the top of Fig. 3.

We now consider thc casc in which some hox at the hottom of Fig. 3 is equivalent to a current source. Let I_b denote the value of this current source (with reference direction chosen to be *out* of the base of the associated transistor). In this case, using the notation of Fig. 1(b), the variahles v_1, i_1, v_2, and i_2, for the associated transistor, are constrained by the relationships:

$$i_1 = \frac{(1 - \alpha_{12}\alpha_{21})f_1(v_1) - \alpha_{12}I_b}{(1 - \alpha_{12})},$$

$$i_2 = \frac{(1 - \alpha_{12}\alpha_{21})f_2(v_2) - \alpha_{21}I_b}{(1 - \alpha_{21})}. \tag{18}$$

Thus, this transistor can be replaced by a pair of diodes (each in series with one of the resistors r_e, r_c) whose nonlinear conductance functions are specified by (18). We may now consider these diodes, these resistors, and the current source, all to be components of the box at the *top* of Fig. 3. We have thus shown, in summary, that when one (or more) of the boxes at the bottom of Fig. 3 is equivalent to a current source, the base-emitter and base-collector voltages of each transistor are still unique, since the network is then equivalent to a network of a type already considered.†

By use of the same type of argument that was applied to the case in which diodes are present in the hox at the top of Fig. 3, the ahove results may, finally, be shown to be valid when diodes are present in the boxes at the bottom of Fig. 3.

The ahove results show the validity of the following statement concerning histable networks: *One cannot synthesize a bistable network which consists of resistors, inductors, capacitors, diodes, independent voltage and current sources, and an arbitrary number of (Fig. 1b) transistors, and which has the structure of Fig. 3 when all capacitors are open-circuited and all inductors are short-circuited.*

APPENDIX

In this appendix we give the proof of a theorem which is used here and which is implied in Ref. 1 hut is not stated explicitly there.

Theorem: If A is a real square matrix then $A \in P_0$ if and only if det $[A + D] \neq 0$ for every diagonal matrix $D > 0$.

† Here, of course, we use the proposition, proved above, that the voltage across each diode in the box at the top of Fig. 3 is unique.

Proof: (*if*) Suppose $A \notin P_0$. If $\det A < 0$ then for sufficiently small $\zeta > 0$, $\det [\zeta I + A] < 0$. For sufficiently large ζ, however,

$$\det [\zeta I + A] = \zeta^n \cdot \det \left[I + \frac{1}{\zeta} A \right] > 0.$$

Thus, since $\det [\zeta I + A]$ is a continuous function of ζ, there exists some value of $\zeta > 0$ such that $\det [\zeta I + A] = 0$. For this value of ζ let $D = \zeta I$.

If $\det A \geqq 0$ but, for some positive integer $k < n$, A has a $k \times k$ principal minor which is negative we may, without loss of generality, assume that A is partitioned as

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_1 is a $k \times k$ matrix with $\det A_1 < 0$. This is so because $\det [D + A]$ is not altered if any two rows and then the corresponding pair of columns are interchanged. Let $D^{(1)} = \operatorname{diag}[d_1, \cdots, d_n]$ with $d_1 = \cdots = d_k = \xi$, where $\xi > 0$ is chosen so small that $\det[\xi I + A_1] < 0$. Then, with $d_{k+1} = \cdots = d_n = \zeta > 0$, we have

$$\det [D^{(1)} + A] = \det \begin{bmatrix} \xi I + A_1 & A_2 \\ A_3 & \zeta I + A_4 \end{bmatrix}$$
$$= \zeta^{n-k} \cdot \det \begin{bmatrix} \xi I + A_1 & A_2 \\ \frac{1}{\zeta} A_3 & I + \frac{1}{\zeta} A_4 \end{bmatrix}.$$

Thus, for $\zeta > 0$ chosen to be sufficiently large, $\det[D^{(1)} + A] < 0$. Now, if $D^{(2)} = \eta I$, for $\eta > 0$, then it is clear that for η chosen sufficiently large,

$$\det [D^{(2)} + A] = \eta^n \cdot \det \left[I + \frac{1}{\eta} A \right] > 0.$$

Thus, if

$$D(\epsilon) = \epsilon D^{(1)} + (1 - \epsilon) D^{(2)},$$

it is clear that there exists a value of ϵ, $0 < \epsilon < 1$, such that $\det [D(\epsilon) + A] = 0$.

(*only if*) By Theorem 1 of Ref. 1, since $A \in P_0$ and $D > 0$, $[D + A] \in P$. Thus, $\det [D + A] \neq 0$. □

REFERENCES

1. Sandberg, I. W. and Willson, A. N., Jr., "Some Theorems on Properties of DC Equations of Nonlinear Networks," B.S.T.J., *48*, No. 1 (January 1969), pp. 1–34.
2. Fiedler, M. and Pták, V., "On Matrices with Non-Positive Off-Diagonal Elements and Positive Principal Minors," Czech. Math. J., *12*, No. 3 (1962), pp. 382–400.
3. Fiedler, M. and Pták, V., "Some Generalizations of Positive Definiteness and Monotonicity," Numer. Math., *9*, No. 2 (1966), pp. 163–172.
4. Sandberg, I. W., "On the Theory of Linear Multi-Loop Feedback Systems," B.S.T.J., *42*, No. 2 (March 1963), pp. 355–382.